Injectivity radii of hyperbolic integer homology 3-spheres

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Abstract. We construct hyperbolic integer homology 3-spheres where the injectivity radius is arbitrarily large for nearly all points of the manifold. As a consequence, there exists a sequence of closed hyperbolic 3-manifolds which Benjamini-Schramm converge to $\mathbb{H}^3$ whose normalized Ray-Singer analytic torsions do *not* converge to the L^2-analytic torsion of $\mathbb{H}^3$. This contrasts with the work of Abert et. al. who showed that Benjamini-Schramm convergence forces convergence of normalized betti numbers. Our results shed light on a conjecture of Bergeron and Venkatesh on the growth of torsion in the homology of arithmetic hyperbolic 3-manifolds, and we give experimental results which support this and related conjectures.

1 Introduction

By Mostow rigidity, a hyperbolic structure on a closed 3-manifold M is unique up to isometry. While the geometry of M is thus completely determined by its underlying topology, it can be difficult to understand the qualitative and quantitative connections between these two facets of M. Here, we show that a geometric property involving injectivity radii can be varied independently of the homology of the manifold. To state our results, we first need some notation. The injectivity radius $\mathrm{inj}_x(M)$ at $x \in M$ is the largest radius for which the ball about x is embedded, and the (lower) injectivity radius of M itself is $\mathrm{inj}(M) = \inf\{\mathrm{inj}_x(M) \mid x \in M\}$. On the topological side, an *integer homology 3-sphere* is a closed 3-manifold M where $H_*(M;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$, and the term *rational homology 3-sphere* is similarly defined. Our main result here is:

1.1 Theorem. *Given positive constants R and ϵ there exists a hyperbolic integer homology 3-sphere M where*

$$\frac{\operatorname{vol}\Big(\big\{x \in M \;\big|\; \operatorname{inj}_x(M) < R\big\}\Big)}{\operatorname{vol}(M)} < \epsilon.$$

In fact, we show that the homology of M can be specified arbitrarily (Theorem 2.1). The proof is based on the modern theory of Kleinian groups; before sketching it, we motivate our result in several ways.

1.2 Cooper's question. Starting with any closed hyperbolic 3-manifold, one can make the injectivity radius arbitrarily large everywhere by taking a suitable finite cover. In the context of the Virtual Haken Conjecture, this motivated Cooper to ask whether there are hyperbolic *rational* homology 3-spheres with arbitrarily large injectivity radius. In fact, such manifolds do exist by the work of Calegari-Dunfield and Boston-Ellenberg [CD1, BE]. However, if one instead considers *integer* homology 3-spheres, then the analogous question is open; our Theorem 1.1 answers affirmatively a weakened version of this question. The manifolds of [CD1, BE] came from a tower of congruence covers of a fixed base manifold, and it seems unlikely this method would work for integer homology 3-spheres as we now describe.

1.3 Torsion growth. Recently, number theorists have become interested in torsion in the homology of arithmetic groups [BV, CV]. Specifically, Bergeron and Venkatesh posited the following as part of an intriguing general conjecture for arithmetic lattices in semisimple Lie groups; in the present context of hyperbolic 3-manifolds, Le independently formulated a closely related conjecture, see [Le] for details.

1.4 Conjecture [BV]. *Let M be a closed congruence arithmetic hyperbolic 3-manifold, and $M \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots$ a tower of congruence covers where* $\operatorname{inj}(M_n) \to \infty$*. Then the size of the torsion subgroup of $H_1(M_n;\mathbb{Z})$ grows exponentially in* $\operatorname{vol}(M_n)$ *and moreover*

$$\lim_{n\to\infty} \frac{\log\left|H_1(M_n;\mathbb{Z})_{\mathrm{torsion}}\right|}{\operatorname{vol}(M_n)} = \frac{1}{6\pi} \tag{1.5}$$

In particular, if this conjecture holds then the approach of [CD1, BE] which used exactly such a tower to answer Cooper's question cannot be modified to prove Theorem 1.1.

One of two key parts to Conjecture 1.4 is the expected convergence of Ray-Singer analytic torsion in such a tower of covers. More precisely, the analytic torsion of a

Riemannian manifold M is

$$\tau(M) = \frac{1}{2} \sum_{k=0}^{\dim M} (-1)^k \cdot k \cdot \log\bigl(\det{}'(\Delta_k)\bigr)$$

where Δ_k is the Laplacian on smooth k-forms. Then for covers M_n as in Conjecture 1.4, part of (1.5) is that one should have

$$\lim_{n\to\infty} \frac{\tau(M_n)}{\mathrm{vol}(M_n)} = \tau^{(2)}(\mathbb{H}^3) \tag{1.6}$$

where $\tau^{(2)}(\mathbb{H}^3) = 1/6\pi$ is the L^2-analytic torsion of $\mathbb{H}^3$. A corollary of Theorem 1.1 is that one *need not* have (1.6) for a sequence M_n of hyperbolic 3-manifolds which Benjamini-Schramm converge to $\mathbb{H}^3$, which is a natural geometric notion of convergence implied by the hypotheses of Conjecture 1.4. As this corollary was the primary motivation for this paper, we now discuss it and its context in detail.

1.7 Benjamini-Schramm convergence. For a manifold M, we define $\mathrm{thin}_R M = \{x \in M \mid \mathrm{inj}_x(M) < R\}$. Following [ABBGNRS], we say that a sequence $\{M_n\}$ of closed hyperbolic 3-manifolds Benjamini-Schramm converge to $\mathbb{H}^3$ if for all $R > 0$ one has $\mathrm{vol}\bigl(\mathrm{thin}_R M_n\bigr)/\mathrm{vol}(M_n) \to 0$ as $n \to \infty$. We emphasize here that the M_n may have no relationship with each other beyond being hyperbolic; in particular, they need not be covers of a fixed manifold. Despite this, Abert et. al. were able to show that this notion of geometric convergence also implies convergence of part of the topology:

1.8 Theorem [ABBGNRS]. *Let M_n be a sequence of closed hyperbolic 3-manifolds which Benjamini-Schramm converge to $\mathbb{H}^3$. Then*

$$\lim_{n\to\infty} \frac{\dim H_1(M_n;\mathbb{Q})}{\mathrm{vol}(M_n)} = 0. \tag{1.9}$$

Here, the 0 in the right-hand side of (1.9) should be interpreted as the first L^2–betti number of $\mathbb{H}^3$, and the moral of Theorem 1.8 is that suitable local convergence of the geometry of the M_n leads to convergence of their normalized betti numbers to the corresponding L^2–betti number of their common universal cover. Theorem 1.8 generalizes results of Lück and Lott [Lück, Lott] which apply only to M_n coming from finite covers of a fixed manifold (as in Conjecture 1.4).

A key consequence of Theorem 1.1 is that Theorem 1.8 does not have an analog for analytic torsion:

1.10 Corollary. *There exist closed hyperbolic 3-manifolds M_n which Benjamini-Schramm converge to $\mathbb{H}^3$ where $\tau(M_n)/\mathrm{vol}(M_n) \to 0$ as $n \to \infty$. In particular, the limit is not $\tau^{(2)}(\mathbb{H}^3) = 1/6\pi$.*

Thus, while the geometric notion of Benjamini-Schramm convergence is enough to control the convergence of (normalized) betti-numbers to the corresponding L^2 invariant of the limit, the same is not true for torsion.

1.11 Experimental results. Corollary 1.10 limits how much one can broaden Conjecture 1.4, and in this narrow sense could be taken as evidence against Conjecture 1.4. However, we present here computational evidence which strongly supports Conjecture 1.4 as well as certain generalizations to nonarithmetic manifolds. Our experiments complement prior work of Şengün [Şen1, Şen2, Şen3] and Page [Pag1]. To frame our results, we need to expand on the connection between Conjecture 1.4 and analytic torsion. For a closed Riemannian 3-manifold, the Cheeger-Müller theorem [Che, Mül] says that

$$\tau(M) = \log\left|H_1(M;\mathbb{Z})_{\mathrm{tor}}\right| - \log\big(\mathrm{vol}(M)\big) + 2\log\big(\text{regulator of } H^1(N)\big) \tag{1.12}$$

Here the regulator of $H^1(N)$ is the covolume of the lattice $H^1(N;\mathbb{Z})$ in $H^1(N;\mathbb{R})$, where the latter has the inner product coming from its identification with the set of harmonic forms. The first part of Conjecture 1.4 is that $\tau(M_n)/\mathrm{vol}(M_n) \to 1/6\pi$ and the second is that $\log\big(\mathrm{reg}\ H^1(M_n)\big)\big/\mathrm{vol}(M_n) \to 0$. In Section 4, we provide evidence in favor of a broadening of the first part Conjecture 1.4 to *all* hyperbolic 3-manifolds:

> **1.13 Conjecture.** *Let M_n be covers of a fixed closed hyperbolic 3-manifold M which Benjamini-Schramm converge to $\mathbb{H}^3$. Then $\tau(M_n)/\mathrm{vol}(M_n) \to 1/6\pi$.*

In contrast, it is not expected that $\log\big(\mathrm{reg}\ H^1(M_n)\big)\big/\mathrm{vol}(M_n) \to 0$ for nonarithmetic manifolds; we give data in support of this, see especially Figure 4.5. For arithmetic manifolds, experiments of Şengün [Şen3] identified the case of congruence covers of prime-power level as a place where such convergence appears to be slowest, to the point where one hits computational limits before getting convincing evidence for or against Conjecture 1.4. In Section 4, we investigate several families of examples of this type. While some of these remain ambiguous, overall they provide additional evidence that $\log\big(\mathrm{reg}\ H^1(M_n)\big)\big/\mathrm{vol}(M_n) \to 0$ even in this case.

1.14 Proof sketch. Given a homeomorphism f of a surface S there are two natural 3-manifolds we can build from it. One is the mapping torus M_f, which fibers over the circle. Alternatively, we can identify S with the boundary of a handlebody H and consider the associated Heegaard splitting: $HS_f = H \cup_f H$. While the natural copies of S in M_f and HS_f are radically different topologically (the first is incompressible and the other maximally compressible), the philosophy of Kleinian groups, specifically [Nam, NS], indicates that in favorable conditions on f, and for large powers n, there are large chunks of the geometry of M_{f^n} and HS_{f^n} that are nearly isometric.

Here is the basic idea behind the manifolds in Theorem 1.1. Fixing $R > 0$, it is easy to construct (S, f) so that M_f has $\mathrm{inj}(M_f) > R+1$. Now for M_f we have $b_1(M_f) > 0$, and in particular M_f is not a homology sphere. However, we will "photocopy" its geometry onto a Heegaard splitting whose homology we can independently control. Specifically, choose homeomorphisms h and g of S so that $HS_h = S^3$ and g acts trivially on $H_1(S;\mathbb{Z})$. Then define M_n to be the Heegaard splitting associated to $h \circ f^n \circ g \circ f^{-n}$. This M_n is an integral homology sphere since the gluing map acts on $H_1(S;\mathbb{Z})$ precisely as h does. We show that f and g can be chosen so that when n is large, most of the geometry of M_n is locally nearly isometric to M_f and hence $\mathrm{inj}_x(M_n) > R$ on most of M_n. Specifically, the volume of $\mathrm{thin}_R M_n$ is uniformly bounded whereas $\mathrm{vol}(M_n) \to \infty$; hence we can make the ratio $\mathrm{vol}(\mathrm{thin}_R M_n)/\mathrm{vol}(M_n) < \epsilon$, as required by Theorem 1.1.

In realizing this outline, there are several different routes one could take through the machinery of Kleinian groups. We choose one which only uses results about manifolds with incompressible boundary and bounded geometry. Moreover, unlike the corresponding parts of [Nam], our argument does not rely on [Tian].

1.15 Open questions. One very natural question is whether there are integral homology 3-spheres where the injectivity radius is large everywhere. From the point of view in the discussion in Sections 1.3 and 1.7, in fact it would be very interesting if one could add to Theorem 1.1 a *uniform* lower bound on $\mathrm{inj}(M)$ independent of R and ϵ. The current construction provides no control on $\mathrm{inj}(M)$ as R varies, basically because the genus of S has to change with R; see Remarks 2.3 and 2.7.

The weaker version of Theorem 1.1 where one just requires that $\mathrm{inj}_x M > R$ for *some* x follows from [PS] by doing $1/n$ Dehn filling on the knot complements constructed there which also have this property. A natural question is whether there are knots in S^3 where $\mathrm{inj}_x M$ is big most places. We give a possible construction of such knots in Remark 2.8.

1.16 Outline of the rest of the paper. Section 2 gives the precise construction of the manifolds in Theorem 1.1 and proves that result modulo the central Lemma 2.6. Section 3 reviews the needed background in Kleinian groups and uses it to prove Lemma 2.6. Finally, Section 4 contains the details of the experimental results.

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2 Proof of the main theorem

The main result of this paper is:

2.1 Theorem. *Given positive constants R and ϵ and a finitely-generated abelian group A, there exists a closed hyperbolic 3-manifold M where*

$$H_1(M;\mathbb{Z}) = A \quad \textit{and} \quad \frac{\operatorname{vol}(\operatorname{thin}_R M)}{\operatorname{vol}(M)} < \epsilon.$$

We begin by constructing a certain 3-manifold which fibers over the circle, the mapping torus of a homeomorphism of a surface, which will be used as the geometric model for most of the manifold in Theorem 2.1.

2.2 Lemma. *Given $R > 0$, there exists a closed hyperbolic 3-manifold M which is a mapping torus where* $\operatorname{inj}(M) > R$*.*

Proof. Fix some hyperbolic mapping torus N. Then N contains finitely many closed geodesics of length $\leq 2R$, corresponding to certain conjugacy classes $[\gamma_i]$ of elements of $\pi_1(N)$. Since $\pi_1(N)$ is residually finite (see e.g. [LR]), there is a finite-index normal subgroup of $\pi_1(N)$ which contains no γ_i. If M is the corresponding finite cover, then its shortest geodesic has length $> 2R$ and hence $\operatorname{inj}(M) > R$. Since the fibration of N over S^1 pulls back to one of M, we are done. □

2.3 Remark. A simple argument using minimal surfaces shows that any mapping torus of a surface of genus g with $\operatorname{inj}(M) = R$ has $\log(g) \geq R - C$, where C is independent of R; thus the genus of the fiber of M in Lemma 2.2 necessarily goes to infinity as R does. While we have no need for this here, with a little more care the above construction can produce examples where $\log(g) \leq 3R + C'$ as we now describe. Specifically, take the base manifold N to be arithmetic of the simplest type, i.e. defined by some quadratic form. (There are many such fibered N by Theorem 5.2 of [Agol].) Now consider a tower M_n of congruence covers of N. If d_n is the degree of $M_n \to N$, by Lemma 2.2.1 of [Yeu] we know there is a constant C'' so that $\operatorname{inj}(M_n) \geq (1/3)\log d_n - C''$. On the other hand, the genus of the fiber grows at most linearly in d_n, and hence satisfies $\log(g) \leq 3R + C'$ for some C' independent of R.

2.4 Main Construction. We now detail the construction of the examples in Theorem 2.1. Throughout, fix $R > 0$ and a finitely generated abelian group A. Via Lemma 2.2, we choose a pseudo-Anosov homeomorphism f of a closed surface S so that the mapping torus M_f has $\operatorname{inj}(M_f) > R + 1$. Let N_0 be a connected sum of lens spaces and copies of $S^2 \times S^1$ so that $H_1(N_0;\mathbb{Z}) = A$. Let g be the genus of S, and let $H^+ \cup H^-$ be a Heegaard splitting of N_0 of genus g; such a splitting exists

provided $g \geq \operatorname{rank}(A)$, and we can always make g bigger if necessary by replacing M_f with a suitable finite cover. Now identify the Heegaard surface $\partial H^+ = \partial H^-$ with S. Choose a pants decomposition P of S so that the pared manifolds $(H^\pm, P)$ are acylindrical; any P which is reasonably far from the disc sets of H^+ and H^- will do, and by Kerckhoff the closure in $\mathscr{PML}(S)$ of all such discs has measure 0 [Ker, Gad].

Let γ be a separating essential simple closed curve on S so that the pared manifold

$$U = \big((S \times [0,2]) \setminus (\gamma \times \{1\}),\ P \times \{0\} \cup P \times \{2\}\big)$$

is acylindrical. We now define a family of links in N_0 which lie in a product neighborhood $S \times [0,6]$ as follows

$$L_n = P \times \{1\} \ \cup\ f^n(P) \times \{2\} \ \cup\ f^n(\gamma) \times \{3\} \ \cup\ f^n(P) \times \{4\} \ \cup\ P \times \{5\}$$

and consider their complements $N_n = N_0 \setminus L_n$. We frame L_n by the blackboard framing with respect to the surfaces $S \times \{s\}$ which contains it; that is, a longitude is a parallel copy of the corresponding component in $S \times \{s\}$. Define the closed manifold $N_{n,k}$ to be the following Dehn surgery on L_n in N_0: do $1/k$ Dehn surgery on each component which is at heights $\{1,2,3\}$ and $-1/k$ Dehn surgery on each component at heights $\{4,5\}$. For large n and k, these $N_{n,k}$ will be the examples used to prove Theorem 2.1. To start, we show

2.5 Lemma. *The homology $H_1(N_{n,k};\mathbb{Z}) = A$ for all n, k.*

Proof. Doing $1/k$ Dehn surgery along a single curve η in S is equivalent to changing the gluing of the Heegaard splitting by the k^{th} power of the Dehn twist on η. Since γ is separating, a Dehn twist on it acts trivially on the homology of S. Thus, homologically, the Dehn twists along the components of L_n at heights $\{1,2\}$ precisely cancel out those at heights $\{4,5\}$. Hence $N_{n,k}$ has the same homology as N_0. □

The key geometric claim is the following, whose proof we defer to Section 3.

2.6 Lemma. *For all large n, the manifold $N_n = N_0 \setminus L_n$ has a complete hyperbolic metric of finite volume, and moreover*

$$\lim_{n\to\infty} \frac{\operatorname{vol}(\operatorname{thin}_R N_n)}{\operatorname{vol}(N_n)} = 0$$

Proof of Theorem 2.1. Let $\epsilon > 0$ be given. By Lemma 2.6, choose n large enough so that N_n is hyperbolic and $\operatorname{vol}(\operatorname{thin}_R N_n)\big/\operatorname{vol}(N_n) < \epsilon/2$. We now view $N_{n,k}$ as a Dehn filling on the cusped manifold N_n. By Thurston's Hyperbolic Dehn Surgery Theorem, for large k the manifold $N_{n,k}$ is hyperbolic; moreover, the geometry of $N_{n,k}$ is

arbitrarily close to that of N_n outside a set of arbitrarily small volume, which is a neighborhood about the core geodesics of the added solid tori [Thu1, PP]. In particular, we can choose k so that $\mathrm{vol}(\mathrm{thin}_R N_{n,k}) \big/ \mathrm{vol}(N_{n,k}) < \epsilon$. Since $H_1(N_{n,k};\mathbb{Z}) = A$ by Lemma 2.5 we have proved the theorem. □

2.7 Remark. For fixed R, the manifolds used to prove Theorem 2.1 can be chosen with minimum injectivity radius bounded below independent of ϵ as we now explain. As shown in Section 3, for large n the manifolds N_n constructed have injectivity radius uniformly bounded below outside neighborhoods of the cusps. Moreover, the geometry of said cusps are nearly isometric for large n. The Drilling Theorem [BB] then shows that the choice of k so that $N_{n,k}$ has geometry close to that of N_n can be made independent of n, and the added core geodesics in $N_{n,k}$ have length uniformly bounded from below.

2.8 Remark. We chose the construction here to streamline the proof of Lemma 2.6 in Section 3. Here is a combinatorially simpler construction satisfying Lemma 2.6 that relies on work of Namazi in his (as yet unpublished) thesis [Nam], the relevant results of which will appear in [BMNS]; we hew to the published literature in our present treatment. Let f be as before, but if necessary change the identification of S with the Heegaard surface of N_0 so that the invariant laminations of f are disjoint from the closure in $\mathscr{PML}(S)$ of the disk sets of both H^+ and H^- (which can be done by [Ker, Gad]). Once again letting γ be a separating curve on S, take N'_n simply to be $N_0 \setminus f^n(\gamma)$. By a bounded geometry model theorem for Heegaard splittings [Nam, BMNS] (similar to Minsky's bounded geometry theorem [Min] in the I-bundle case), given a sufficiently large k, chosen independent of n, the geometry of a $1/k$ Dehn-filling of N'_n will be modelled up to bi-Lipschitz distortion by the geometry of that of M_f for almost all of its volume. An exactly analogous argument to the one given in the proof of Theorem 3.5 allows us to make the bi-Lipschitz constant arbitrarily close to 1 for almost all of the volume. In our current treatment, the extra pairs of pants used to define N_n give us many canonical thrice-punctured spheres which, because of their rigidity, are natural places from which to understand the overall geometry of N_n via geometric limits.

3 Proof of the main lemma

The proof of Lemma 2.6 is our point of entry into the modern theory of Kleinian groups. We first isolate the necessary background before turning to the proof itself.

3.1 Kleinian background. Throughout Section 3, we take S to be a closed surface of genus $g > 1$. We denote by $AH(S)$ the set of all complete hyperbolic 3-manifolds

$M = \mathbb{H}^3/\Gamma$ equipped with *markings*, or homotopy equivalences $h\colon S \to M$, up to marking preserving isometry; precisely,

$$(h\colon S \to M) \sim (g\colon S \to N)$$

if there is an isometry $\phi\colon M \to N$ where $\phi \circ h \simeq g$. The *mapping class group* $\mathcal{MCG}(S)$ of orientation preserving self-homeomorphisms of S up to isotopy acts on $AH(S)$ by precomposition: given $f \in \mathcal{MCG}(S)$ we let

$$f \cdot (h\colon S \to M) = \left(h \circ f^{-1}\colon S \to M\right).$$

We refer to this action as *remarking* the element $(h\colon S \to M)$ by f.

A hyperbolic 3-manifold M determines a conjugacy class of *Kleinian groups*, that is, of discrete subgroups of $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2\mathbb{C}$. A specific group is identified by a choice ω of a *baseframe*, that is, an orthonormal frame in the tangent space at some point in M; the group Γ is then taken so that the covering

$$\mathbb{H}^3 \to M = \mathbb{H}^3/\Gamma$$

sends the standard baseframe $\widetilde{\omega}$ at the origin in $\mathbb{H}^3$ to ω. In practice, we will refer to a base-frame ω as being *in* M in reference to the underlying basepoint.

Via baseframes, $AH(S)$ is readily seen to be the set of conjugacy classes of discrete faithful representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, via the association $[\rho] = h_*$. A sequence (h_n, M_n) of elements of $AH(S)$ *converges algebraically* to (h, M) if there are smooth homotopy equivalences $\varphi_n\colon M \to M_n$ with $\varphi_n \circ h \simeq h_n$ that converge in the C^∞-topology to a *local* isometry.

If a baseframe ω in M is chosen so that (M, ω) has corresponding Kleinian group Γ, we can reformulate algebraic convergence: there are baseframes ω_n in M_n for which the corresponding Kleinian groups Γ_n admit isomorphisms $\rho_n\colon \pi_1(S) \to \Gamma_n$ which converge to $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ in the sense that $\rho_n(\gamma) \to \rho(\gamma)$ for all $\gamma \in \pi_1(S)$; here $\rho_n = (h_n)_*$ and $\rho = h_*$. Thus, the choice of baseframe pins down the representative of each conjugacy class of representations.

Based manifolds (M_n, ω_n) *converge geometrically* to a *geometric limit* (M_G, ω_G) if their associated Kleinian groups Γ_n converge to the associated Kleinian group Γ for (M_G, ω_G) in the *Hausdorff topology*:

(a) for each $\gamma \in \Gamma$ there are $\gamma_n \in \Gamma_n$ so that $\gamma_n \to \gamma$, and

(b) the limit in $\mathrm{PSL}_2\mathbb{C}$ of any convergent subsequence of γ_n lies in Γ.

By elementary compactness results (see [McM, Prop. 2.1]), any algebraically convergent sequence $(h_n, M_n) \to (h, M)$ has a subsequence with an associated geometric

limit M_G; this geometric limit is obtained by choosing baseframes ω_n to obtain convergent representations $\rho_n \to \rho$ and then passing to a convergent subsequence of the corresponding sequence of based manifolds (M_n, ω_n).

Note that we have a locally isometric covering map $(M, \omega) \to (M_G, \omega_G)$. The sequence (h_n, M_n) *converges strongly* if it converges both algebraically and geometrically and moreover the locally isometric cover $M \to M_G$ is an isometry (in particular, a *homeomorphism*).

Geometric convergence also has this intrinsic formulation: $(M_n, \omega_n) \to (M_G, \omega_G)$ if for each compact subset $K \subset M_G$ with $\omega_G \in K$, there are bi-Lipschitz embeddings

$$\psi_n \colon (K, \omega_G) \to (M_n, \omega_n)$$

for n sufficiently large so that ψ_n converge to isometries in the C^∞-topology. While the limit (M_G, ω_G) depends on the choice of baseframes ω_n, if ω'_n lie at a uniformly bounded distance from ω_n then any limit of the sequence (M_n, ω'_n) is isometric to M_G. We remark that when ω_n and ω have been chosen so that we have convergence

$$\rho_n = (h_n \colon S \to (M_n, \omega_n))_* \to \rho = (h \colon S \to (M, \omega))_*$$

on generators, the images $h_n(S)$ sit at uniformly bounded distance from the baseframes ω_n.

3.2 Maximal Cusps. If P and Q are sets of simple closed curves giving a pants decomposition of S, denote by $M(P, Q)$ the corresponding pared manifold

$$\big(S \times I,\ P \times \{0\} \cup Q \times \{1\}\big).$$

We say $M(P, Q)$ is *pared acylindrical* if no simple closed curve isotopic into P is also isotopic into Q. For pared acylindrical $M(P, Q)$ there is a finite-volume hyperbolic structure on $S \times \mathbb{R}$ so that each free homotopy class represented by the pared locus corresponds to a rank-1 cusp. The hyperbolic structure is unique, and letting S mark $M(P, Q)$ by its inclusion as $S \times \{1/2\}$, we obtain a boundary point in the deformation space $AH(S)$ known as a *maximal cusp*.

The convex core of $M = \mathbb{H}^3/\Gamma$, denoted $\mathrm{core}(M)$, is the quotient by Γ of the smallest convex subset of $\mathbb{H}^3$ whose closure contains the limit set of Γ, which is the intersection of the closure of an orbit of Γ with $\hat{\mathbb{C}} = S^2_\infty$. The pared convex core, written $\mathrm{core}^0(M)$, is the complement in $\mathrm{core}(M)$ of its intersection with the Margulis thin parts of M corresponding to cusps. While $\mathrm{core}\big(M(P, Q)\big)$ has frontier consisting of totally geodesic triply-punctured spheres, the boundary of $\mathrm{core}^0\big(M(P, Q)\big)$ consists of a pair of compact surfaces each containing a collection of distinguished annuli representing its intersection with cusps corresponding to P and Q respectively.

Much of the theory of algebraic and geometric limits of quasi-Fuchsian manifolds $Q(X,Y)$ in $AH(S)$ can be carried out for maximal cusps $M(P,Q)$ by viewing the pair (P,Q) as a combinatorial version of the pair $(X,Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ of marked conformal structures determining $Q(X,Y)$. Indeed, as each $M(P,Q)$ is uniquely determined by the choice of P and Q, much of the theory becomes more concrete in this setting.

3.3 Pseudo-Anosov double limits. For a pseudo-Anosov element $f \in \mathcal{MCG}(S)$, we fix a fiber F in the associated mapping torus M_f, the corresponding fibration over S^1 with monodromy f. We define the *block* B_f of f to be M_f split open along F, that is, the closure of $M_f \setminus F$ in the path metric. We define $\widetilde{M}_f$ to be the infinite-cyclic cover of M_f corresponding to $\pi_1(F)$.

Thurston and McMullen showed that the double iteration $Q\big(f^{-n}(X), f^n(X)\big)$ of f on quasi-Fuchsian manifolds converges strongly to $\widetilde{M}_f$. Likewise, McMullen established that the one-sided iteration $Q\big(X, f^n(X)\big)$ converges strongly to a limit Q_f with one end asymptotically isometric to $\widetilde{M}_f$: there is a bi-Lipschitz diffeomorphism between neighborhoods of the infinite-volume end of $\mathrm{core}(Q_f)$ and an end of $\widetilde{M}_f$ so that the norm of the derivative converges to 1. Each of these discussions can be carried out in the setting of maximal cusps:

3.4 Proposition. *The maximal cusps $M\big(f^{-m}(P), f^n(P)\big)$ for $m,n > 0$ converge strongly to $\widetilde{M}_f$ as $m,n \to \infty$. The one-sided iteration $M\big(P, f^n(P)\big)$ converges strongly to a manifold M_A whose pared convex core contains one compact boundary surface S with parabolic locus P and a degenerate end asymptotically isometric to the positive end of $\widetilde{M}_f$. The analogous statement holds for $M\big(f^{-n}(P), P\big)$, whose limit is denoted M_C.*

See Figure 3.7 for schematic pictures of M_A and M_C.

Proof sketch. There are various ways to deduce these results, which follow easily from variations of the original arguments in [Thu2, McM]. Perhaps the simplest is the following, where for concreteness we focus on the first claim. Consider a surface $X \in \mathrm{Teich}(S)$ where P has very short total length and apply the Drilling Theorem of [BB] to the short geodesic representatives of $f^{-m}(P)$ and $f^n(P)$ in the quasi-Fuchsian hyperbolic 3-manifold $Q_{m,n} = Q\big(f^{-m}(X), f^n(X)\big)$. The drilled manifold $D_{m,n}$ has a bi-Lipschitz diffeomorphism between $\mathrm{core}^0(D_{m,n})$ and a subset of $Q_{m,n}$; this diffeomorphism can be made arbitrarily close to isometric by making the length of P on X small enough. Now since $D_{m,n}$ has a cover isometric to $M\big(f^{-m}(P), f^n(P)\big)$, a diagonal argument yields the proposition. □

Our main result of this section is:

3.5 Theorem. *Given a pseudo-Anosov $f \in \mathcal{MCG}(S)$ and a pants decomposition P of S, let $Y_n = M\big(f^{-n}(P), f^n(P)\big)$. For each $\epsilon > 0$ there are finite-volume hyperbolic 3-manifolds A and C so that for all n sufficiently large,* $\mathrm{core}(Y_n)$ *has a decomposition*

$$\mathrm{core}(Y_n) = A_n \cup B_n \cup C_n$$

where A_n and C_n are $1+\epsilon$ bi-Lipschitz to A and C and $\mathrm{inj}_b(Y_n) > \mathrm{inj}(M_f) - \epsilon$ *for every $b \in B_n$. Moreover* $\mathrm{vol}(B_n) \to \infty$ *as $n \to \infty$.*

3.6 Remark. The theory of Kleinian surface groups provides considerable information about the manifolds Y_n; in particular, Minsky's Bounded Geometry Theorem [Min] guarantees there is a *bi-Lipschitz model* for $\mathrm{core}^0(Y_n)$ which can be described as a union of finitely many copies of B_f, and the bi-Lipschitz constant depends only on the genus of the fiber F (we give a more detailed discussion in the proof of Theorem 3.5). Because we wish to ensure that the injectivity radius on B_n is large, the dependence of the bi-Lipschitz constant on the genus presents a difficulty, as the lower bound for the injectivity radius of M_f also depends on the genus of F. Nevertheless we use this bi-Lipschitz control as a starting point.

Before proving Theorem 3.5, we explain its connection to the geometry of the manifolds N_n from Section 2.4 and how it proves Lemma 2.6.

Proof of Lemma 2.6. We return to the notation from Section 2.4. Let $M^\pm$ be the convex cores of the manifolds corresponding to the pared manifolds $(H^\pm, P)$. Let D be the convex core of the hyperbolic manifold corresponding to U, and D_n be its remarking by f^n, i.e. D_n be the convex core of the pared manifold

$$U_n = \big((S\times I)\setminus(f^n(\gamma)\times\{1/2\}),\ f^n(P),\ f^n(P)\big)$$

Then N_n is the union of the following pieces, glued along their totally geodesic surface boundaries (since these are all thrice-punctured spheres there are no moduli issues):

$$N_n = M^+ \cup \mathrm{core}\big(M(P, f^n(P))\big) \cup D_n \cup \mathrm{core}\big(M(f(P^n), P)\big) \cup M^-$$

The geometries of $M^\pm$ and D_n are fixed, and in particular so are their volumes. For large n, by Theorem 3.5, the other pieces have injectivity radius at least $\mathrm{inj}(M_f) - \epsilon$ outside a set of uniformly bounded volume. This proves Lemma 2.6. □

Proof of Theorem 3.5. The mapping torus M_f is defined as $S \times [0,1]$ where $(x,1) \sim (f(x), 0)$. The cover $\widetilde{M}_f$ is thus $S \times \mathbb{R}$ where the deck group is generated by the self-isometry α sending (x,t) to $\big(f^{-1}(x), t+1\big)$. We take our preferred fiber F in M_f to be

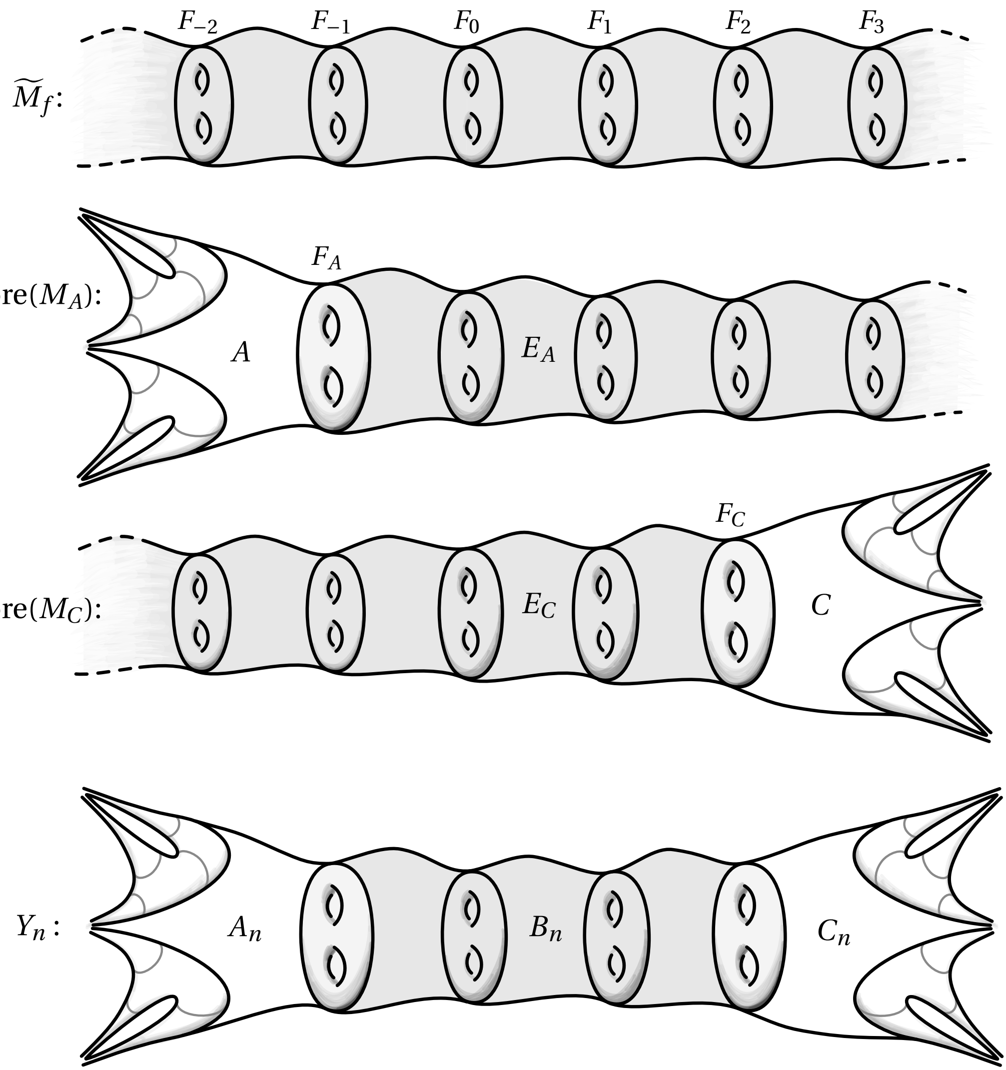


Figure 3.7. The manifolds used in the proof of Theorem 3.5.

$S \times \{0\}$, and the default marking $h_0 \colon S \to \widetilde{M}_f$ to be the inclusion of S as $S \times \{0\}$. Note that the action of f on $AH(S)$ commutes with the action by α:

$$\alpha \circ h_0 \simeq f \cdot h_0 = h_0 \circ f^{-1}.$$

Further, we denote by F_k the translate $\alpha^k(F) = S \times \{k\}$ of the fiber; compare the top of Figure 3.7. For $k < k'$ we denote by $[F_k, F_{k'}]$ the compact submanifold of $\widetilde{M}_f$ which is the complement of the open infinite-volume components of $\widetilde{M}_f \setminus (F_k \cup F_{k'})$.

We may consider the marking $h_k \colon S \to \widetilde{M}_f$ where

$$h_k = \alpha^k \circ h_0 \colon S \to \widetilde{M}_f.$$

Here, $h_k(S)$ is F_k and as elements of $AH(S)$ we have

$$(h_k, \widetilde{M}_f) = f^k(h_0, \widetilde{M}_f).$$

By the Bounded Geometry Theorem [Min], there is an L depending only on S so that for all large n the manifold $\mathrm{core}^0(Y_n)$ admits an L–bi-Lipschitz homeomorphism, or *model map*,

$$\phi_n\colon [F_{-n}, F_n] \to \mathrm{core}^0(Y_n).$$

Since the volume of $[F_{-n}, F_n]$ is $2n\,\mathrm{vol}(M_f)$, we have

$$\mathrm{vol}\big(\mathrm{core}^0(Y_n)\big) \to \infty \quad \text{as } n \to \infty.$$

The homotopy class of ϕ_n is chosen so that $\phi_n \circ h_0$ corresponds to the standard marking on Y_n; in other words, as elements of $AH(S)$ we have

$$(\phi_n \circ h_0, Y_n) = M\big(f^{-n}(P), f^n(P)\big).$$

For each integer k with $|k| < n$, the copy of the fiber F_k provides a marking for Y_n via the model map ϕ_n by taking

$$\phi_n \circ h_k\colon S \to Y_n,$$

marked by the translate F_k in $[F_{-n}, F_n]$. Then we have

$$(\phi_n \circ h_k, Y_n) = f^k(\phi_n \circ h_0, Y_n).$$

Let

$$g_{n,k} = \phi_n \circ h_k$$

denote this marking, and note that $g_{n,0}$ corresponds to the standard marking of Y_n.

We note that for each k with $|k| \le n$, the manifold $M\big(f^{-n+k}(P), f^{n+k}\big)$ is isometric to $M\big(f^{-n}(P), f^n(P)\big) = Y_n$. In particular, indexing the one-sided iterations by $M\big(P, f^{2n}(P)\big)$ and $M\big(f^{-2n}(P), P\big)$ we obtain manifolds that are isometric to Y_n by the isometry α^n and α^{-n} respectively.

To prove the theorem, we start by describing A_n and C_n. By Proposition 3.4, the sequences $\{M\left(P, f^{2n}(P)\right)\}$ and $\{M\left(f^{-2n}(P), (P)\right)\}$ converge strongly to limits in $AH(S)$ with one end asymptotically isometric to the positive end of $\widetilde{M}_f$ and the negative end of $\widetilde{M}_f$ respectively. The sequence $\{M\left(f^{-n}(P), f^n(P)\right)\}$ converges strongly to $\widetilde{M}_f$ itself.

Let M_A in $AH(S)$ be the strong limit of $M\big(P, f^{2n}(P)\big)$. We now explain the needed decomposition of M_A which is sketched in Figure 3.7. By Proposition 3.4 there is an embedded surface F_A in $\mathrm{core}(M_A)$, homotopic to the marking, so that F_A divides $\mathrm{core}(M_A)$ into a component A with bounded volume and an infinite-volume

(neighborhood of an) end E_A so that E_A is $1+\epsilon/(2\,\mathrm{inj}(M_f))$ bi-Lipschitz to (a neighborhood of) the positive end of $\widetilde{M}_f$. The finite-volume submanifold $A \subset \mathrm{core}(M_A)$ has boundary

$$\partial A = \partial\mathrm{core}(M_A) \sqcup F_A.$$

In particular, A is chosen so that we have

$$\mathrm{inj}_b(M_A) > \mathrm{inj}(\widetilde{M}_f) - \epsilon/2 \quad \text{for each } b \in E_A. \tag{3.8}$$

We take C to be the analogous subset of M_C, the limit of $M\big(f^{-2n}(P), P\big)$ in $AS(S)$, cut off by a surface F_C; see Figure 3.7.

The intersection $A^0 = \mathrm{core}^0(M_A) \cap A$ being compact, the strong convergence of $M\big(P, f^{2n}(P)\big)$ to M_A guarantees, for n sufficiently large, smooth bi-Lipschitz embeddings

$$\psi_{2n}\colon A^0 \to M\big(P, f^{2n}(P)\big)$$

converging C^∞ to an isometry. We let A_n be the bounded volume submanifold of $M\big(P, f^{2n}(P)\big)$, which is isometric to Y_n, cut off by the image $\psi_{2n}(F_A)$ and the convex core boundary components corresponding to the negative end of $M\big(P, f^{2n}(P)\big)$; compare Figure 3.7. We define C_n similarly and take

$$B_n = \mathrm{core}(Y_n) \setminus (A_n \cup C_n).$$

Since $\mathrm{vol}\big(\mathrm{core}(Y_n)\big)$ goes to infinity whereas $\mathrm{vol}(A_n)$ and $\mathrm{vol}(C_n)$ are uniformly bounded, it follows that $\mathrm{vol}(B_n) \to \infty$ as $n \to \infty$, verifying the last sentence of Theorem 3.5.

We now show that for n sufficiently large we have

$$\mathrm{inj}(B_n) > \mathrm{inj}(\widetilde{M}_f) - \epsilon.$$

Assume otherwise, and let p_n be a sequence of points in B_n for which

$$\mathrm{inj}_{p_n}(Y_n) \le \mathrm{inj}(\widetilde{M}_f) - \epsilon. \tag{3.9}$$

Then by the uniform density of the fibers F_k in $[F_{-n}, F_n]$ the L–bi-Lipschitz model map

$$\phi_n\colon [F_{-n}, F_n] \to \mathrm{core}^0(Y_n)$$

guarantees there is a sequence $\{k_n\}$ with $|k_n| < n$ so that p_n lies at distance at most $L \cdot \mathrm{diam}(B_f)$ from the image $\phi_n(F_{k_n}) = g_{n,k_n}(S)$.

The sequence (g_{n,k_n}, Y_n) in $AH(S)$ is represented by remarking Y_n by f^{k_n}. Said differently, in $AH(S)$ we have

$$(g_{n,k_n}, Y_n) = f^{k_n}(g_{n,0}, Y_n)$$

and $(g_{n,0}, Y_n)$ represents the standard marking for which

$$(g_{n,0}, Y_n) = M\big(f^{-n}(P), f^n(P)\big).$$

Since the basepoints p_n lie at a uniformly bounded distance from the marking surfaces $g_{n,k_n}(S)$, we may study the injectivity radii at p_n in terms of the limiting geometry of

$$(g_{n,k_n}, Y_n) = M\big(f^{k_n-n}(P), f^{k_n+n}(P)\big).$$

Our analysis breaks into two cases, depending on whether $n-|k_n|$ is bounded.

Case $n-|k_n|$ is unbounded. After passing to a subsequence where $n-|k_n| \to \infty$, Proposition 3.4 gives that the sequence (g_{n,k_n}, Y_n) converges strongly to $\widetilde{M}_f$. As each p_n lies within $L \cdot \mathrm{diam}(B_f)$ of the marking $g_{n,k_n}(S)$ there is a compact subset $K \subset \widetilde{M}_f$ and smooth embeddings $\psi_n \colon K \to Y_n$ converging C^∞ to an isometry so that $p_n \in \psi_n(K)$. It follows that $\mathrm{inj}_{p_n}(Y_n) > \mathrm{inj}(\widetilde{M}_f) - \epsilon$ for n sufficiently large contradicting assumption (3.9).

Case $n-|k_n|$ is bounded. We first pass to a subsequence where one of $n-k_n$ and $-n-k_n$ is bounded; for notational simplicity we suppose $|-n-k_n| < d$. Then the basepoint p_n lies within a uniformly bounded distance, namely $D = d \cdot L \cdot \mathrm{diam}(B_f)$, of the marking surface $g_{n,-n}(S)$.

We now employ the strong convergence of $M\big(P, f^{2n}(P)\big)$ to M_A. Let $K \cong F_A \times [-1,1]$ denote a compact product neighborhood of F_A in M_A containing the ball $B_{2D}(A^0)$. By strong convergence, we have bi-Lipschitz embeddings $\psi_n \colon K \to Y_n$ that send the neighborhood K of F_A to a neighborhood of the image $\psi_n(F_A) \subset \partial A_n$ by an orientation-preserving diffeomorphism. For n sufficiently large, the embeddings ψ_n extend to diffeomorphisms on all of M_A; in particular, the preimages $\psi_n^{-1}(B_n)$ of the subsets B_n lie in the positive end E_A of M_A.

Now as each p_n lies within distance D of $g_{n,-n}(S)$ and the latter is contained in $\psi_n(A^0)$, it follows that p_n lies in $\psi_n(K)$ for all large n. Our basepoints p_n are in B_n and hence as discussed we have that $\psi_n^{-1}(p_n)$ lies in E_A. Now by (3.8) the injectivity radius of E_A is at least $\mathrm{inj}(\widetilde{M}_f) - \epsilon/2$. Thus for large n we must have $\mathrm{inj}_{p_n}(Y_n) > \mathrm{inj}(\widetilde{M}_f) - \epsilon$ which again contradicts assumption (3.9).

This shows that for sufficiently large n we have $\mathrm{inj}_b(Y_n) > \mathrm{inj}(M_f) - \epsilon$ for every $b \in B_n$, completing the proof of Theorem 3.5. $\square$

4 Experimental results

For a finite-volume hyperbolic 3-manifold (or 3-orbifold), define

$$\mathrm{TorRat}(M) = \frac{6\pi \cdot \log \big| H_1(M;\mathbb{Z})_{\mathrm{torsion}} \big|}{\mathrm{vol}(M)}$$

Conjecture 1.4 is then that $\mathrm{TorRat}(M_n) \to 1$ for a suitable tower M_n of congruence covers of a fixed arithmetic manifold. As discussed in Section 1.11, part of this conjecture is that $\tau(M_n) \to 1/6\pi$. Given the Cheeger-Müller formula (1.12), it makes sense to slightly modify our definition here to

$$\mathrm{TorRat}(M) = 6\pi \cdot \left(\frac{\log\left|H_1(M;\mathbb{Z})_{\mathrm{tor}}\right|}{\mathrm{vol}(M)} - \frac{\log\left(\mathrm{vol}(M)\right)}{\mathrm{vol}(M)} \right)$$

While these two definitions are asymptotically the same as $\mathrm{vol}(M) \to \infty$, the revised one has the feature that when $b_1(M) = 0$ we have $\mathrm{TorRat}(M) = 6\pi \cdot \tau(M)$. In particular, Conjecture 1.13 in the case where all $b_1(M_n) = 0$ is equivalent to $\mathrm{TorRat}(M_n) \to 1$.

4.1 Twist-knot orbifolds. First, we consider the 34 hyperbolic 3-orbifolds of Section 7 of [CD1]. These are topologically similar in that they are all built from twist-knots, but some are arithmetic and others are not. As in [CD1], we consider Γ_0–type congruence covers of prime level, and explore what happens to $\mathrm{TorRat}(M)$ in these covers.

Let us start with the 11 twist-knot orbifolds which are arithmetic. Going through prime levels of norm in [500, 15,000] gave some 14,990 congruence covers of Γ_0–type, which are plotted in Figure 4.4; as with the experiments of [Şen1, Pag1], this data is very consistent with Conjecture 1.4. Notice in Figure 4.4 that the red dots ($b_1 > 0$) appear to be somewhat lower (on average) than the blue dots ($b_1 = 0$). To confirm this, we focus on the tail of 2,253 covers where $\mathrm{vol}(M) > 15{,}000$ and plot the distribution of TorRat for both types; see Figure 4.6. This pattern is expected since when $b_1(M) > 0$ the analytic torsion $\tau(M)$ gets a contribution from the regulator of $H^1(M)$; thus even if $\tau(M) \approx 1$ then $\mathrm{TorRat}(M)$ can be noticeably less than 1. Figure 4.8 further explores the effect the size of b_1 on TorRat.

Next, we consider the 23 twist-knot orbifolds which are nonarithmetic. In this case, there are some 31,391 congruence covers of this type, which are plotted in Figure 4.5. Two things are worth pointing out here. The first is that when $b_1(M) = 0$ one continues to have $\mathrm{TorRat}(M) \to 1$ as $\mathrm{vol}(M) \to \infty$, which is strong evidence for Conjecture 1.13 and also consistent with the nonarithmetic examples of [Şen2]. Surprisingly, the convergence of $\mathrm{TorRat}(M) \to 1$ appears to be *faster* than in the arithmetic case, as shown in Figure 4.7. The second thing is that when $b_1(M) > 0$ there are examples where $\mathrm{TorRat}(M)$ is much less than 1 even when the $\mathrm{vol}(M)$ is quite large; this suggests that Conjecture 1.4 cannot be broadened to nonarithmetic manifolds. A more detailed look at the effect of b_1 on TorRat is given in Figure 4.9.

4.2 Covers of prime-power level. In the case of Bianchi manifolds, Şengün [Şen3] discovered that for congruence covers of the form $\Gamma_0(\mathfrak{p}^n)$ where $\mathfrak{p}$ is a prime of small norm, then TorRat is much smaller than in the prime-level case for covers of similar

volume. In particular, one hits a computational wall before getting convincing evidence that TorRat $\to 1$. Here, we look at several closed arithmetic examples which exhibit the same phenomenon; in one case, we are able find a cover with TorRat ≈ 1 providing further evidence for Conjecture 1.4. Part of the issue here is that these examples can have a lot of $b_1(M)$ and hence potentially a large contribution to $\tau(M)$ from the regulator of $H^1(M)$.

In order to tease apart the issues here, we start with some families where $b_1(M) = 0$ for all the covers and hence $\mathrm{TorRat}(M) = 6\pi \cdot \tau(M)$. Section 6.7 of [CD1] gives 19 closed hyperbolic 3-manifolds (of which 3 are arithmetic) where there is a prime $\mathfrak{p}$ of norm 2 where the associated quaternion algebra ramifies and moreover where $\pi_1(M)$ is 2-powerful. Consequently, by Theorem 6.3 of [CD1] the congruence covers of level $\mathfrak{p}^n$ all have $b_1(M) = 0$. The data on 68 covers of these manifolds is shown in Figure 4.10. The convergence of TorRat to 1 seems reasonably convincing; for the 12 covers with volumes > 15,000, the values of TorRat are in $[1.000, 1.125]$. This is still slower than the convergence observed for covers of prime level, especially considering that most of the manifolds here are nonarithmetic; compare Figure 4.7. Another arithmetic example whose $\Gamma_0(\mathfrak{p}^n)$–covers have $b_1 = 0$ for a prime of norm 2 is given in Figure 4.11; this example has the best convergence of any tower of prime-power level that we found. Some additional data for other arithmetic manifolds and primes of norm 5 where again $b_1 = 0$ is given in Figure 4.12.

We turn now to five families of examples where the $\Gamma_0(\mathfrak{p}^n)$–covers have $b_1 > 0$ and hence the regulator term of TorRat comes into play. In each case, we start with the arithmetic base orbifold coming from the elements of norm one in a maximal order of a quaternion algebra D over a field K. The quaternion algebra D is ramified at all the real places of K and at finitely many primes of K as specified in Figure 4.13. Figure 4.13 shows a marked correlation between the amount of b_1 and how close TorRat is to 1. While the data is not completely conclusive, except perhaps in the case of M_1, it is consistent with the conjecture that TorRat $\to 1$.

4.3 Computational notes. The computations here were done with Magma [CBFS]. The code for building the covers of twist-knot orbifolds is available at [CD2]. The base orbifolds for Section 4.2 were constructed by Page's program KleinianGroups [Pag2].

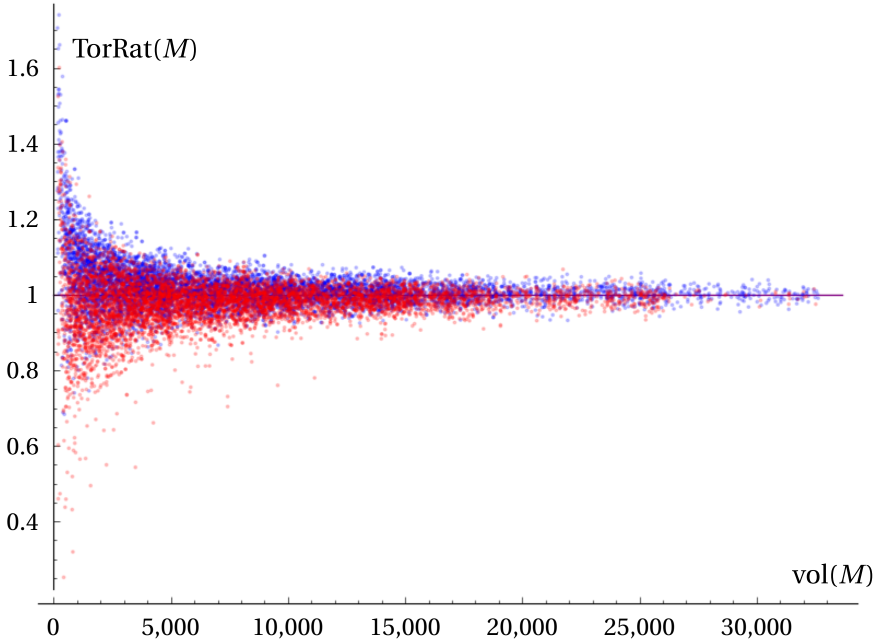


Figure 4.4. Congruence covers of arithmetic twist-knot orbifolds. The blue dots are covers where $b_1 = 0$ and the red dots covers where $b_1 > 0$.

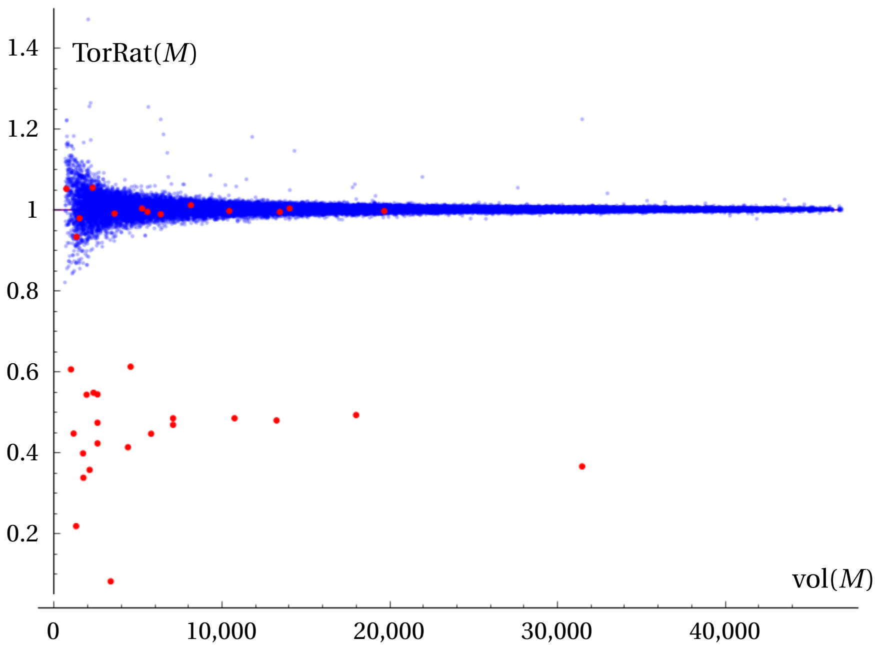


Figure 4.5. Congruence covers of nonarithmetic twist-knot orbifolds; as before, blue dots indicate $b_1 = 0$ and red dots $b_1 > 0$.

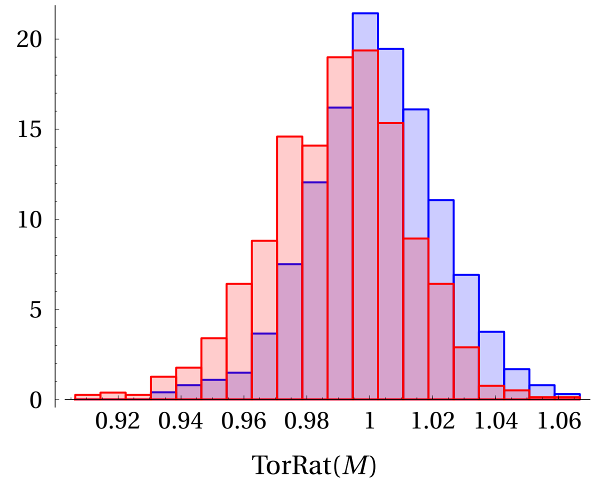


Figure 4.6. Histogram for TorRat(M) for arithmetic covers of twist-knot orbifolds with vol(M) > 15,000; as before, red is $b_1 > 0$ and blue $b_1 = 0$.

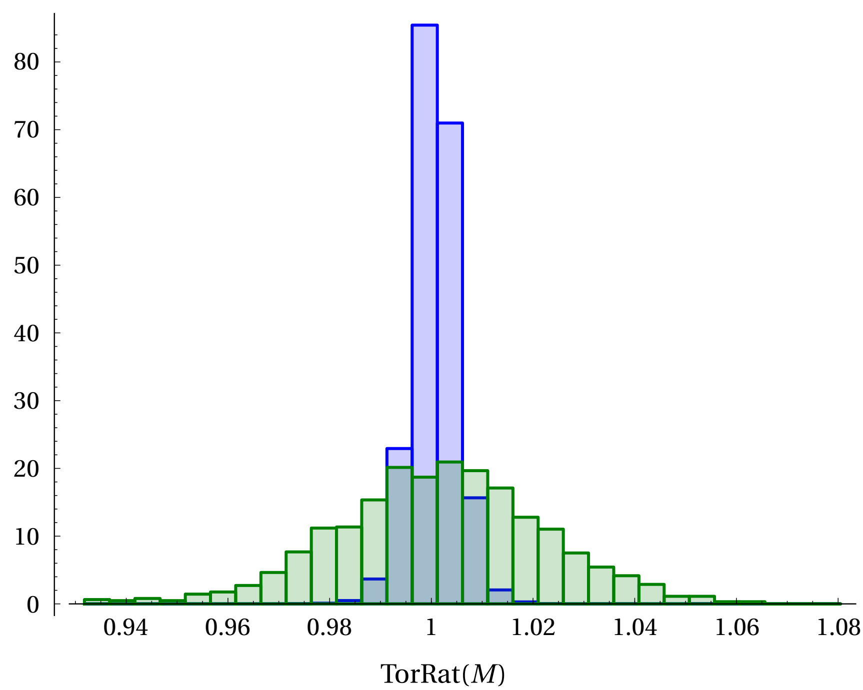


Figure 4.7. Histograms for covers where vol(M) > 15,000. In blue are all the nonarithmetic covers (with two outliers removed), and in green are arithmetic covers with $b_1 = 0$.

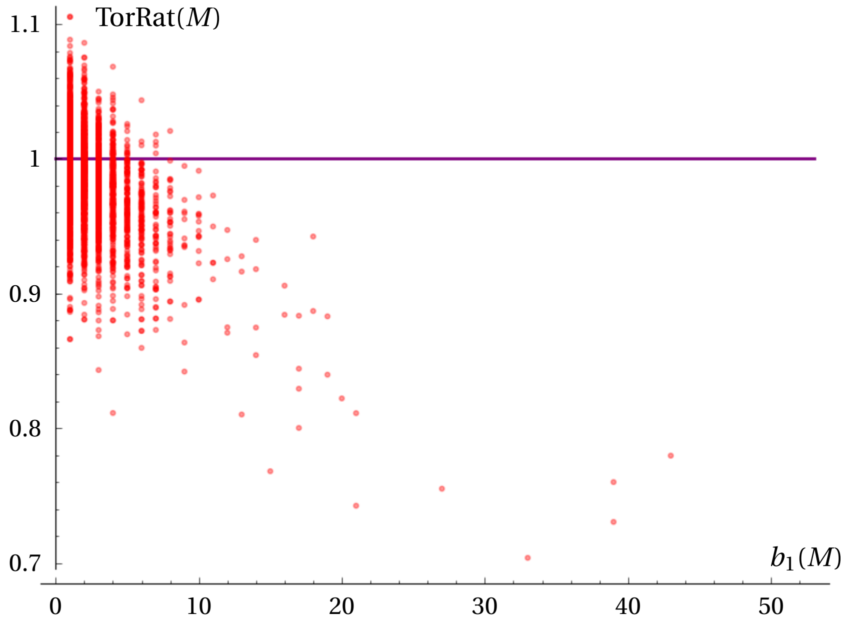


Figure 4.8. The relationship between TorRat(M) and $b_1(M)$ for covers of arithmetic twist-knot orbifolds where $b_1(M) > 0$. Excludes covers of volume less than 5,000.

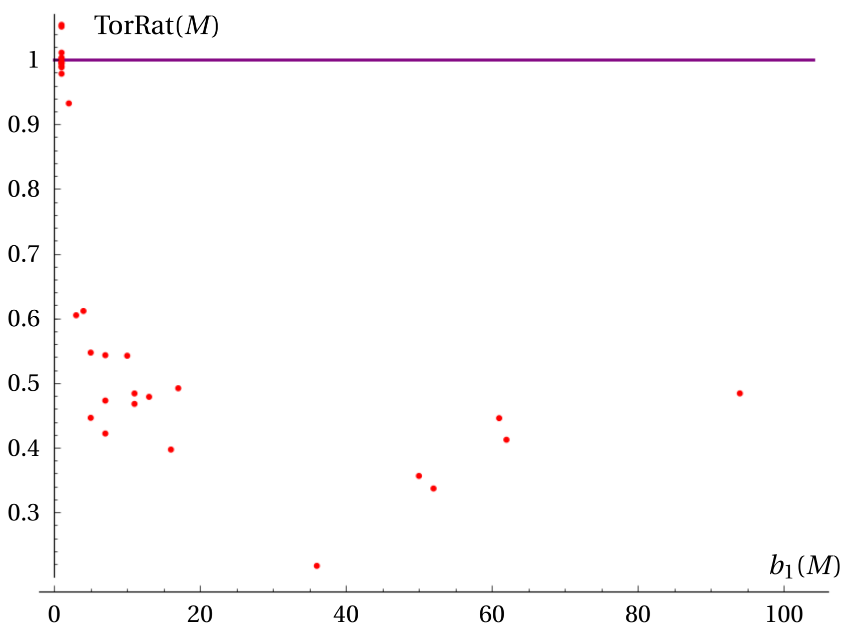


Figure 4.9. Covers of nonarithmetic twist-knot orbifolds with $b_1 > 0$.

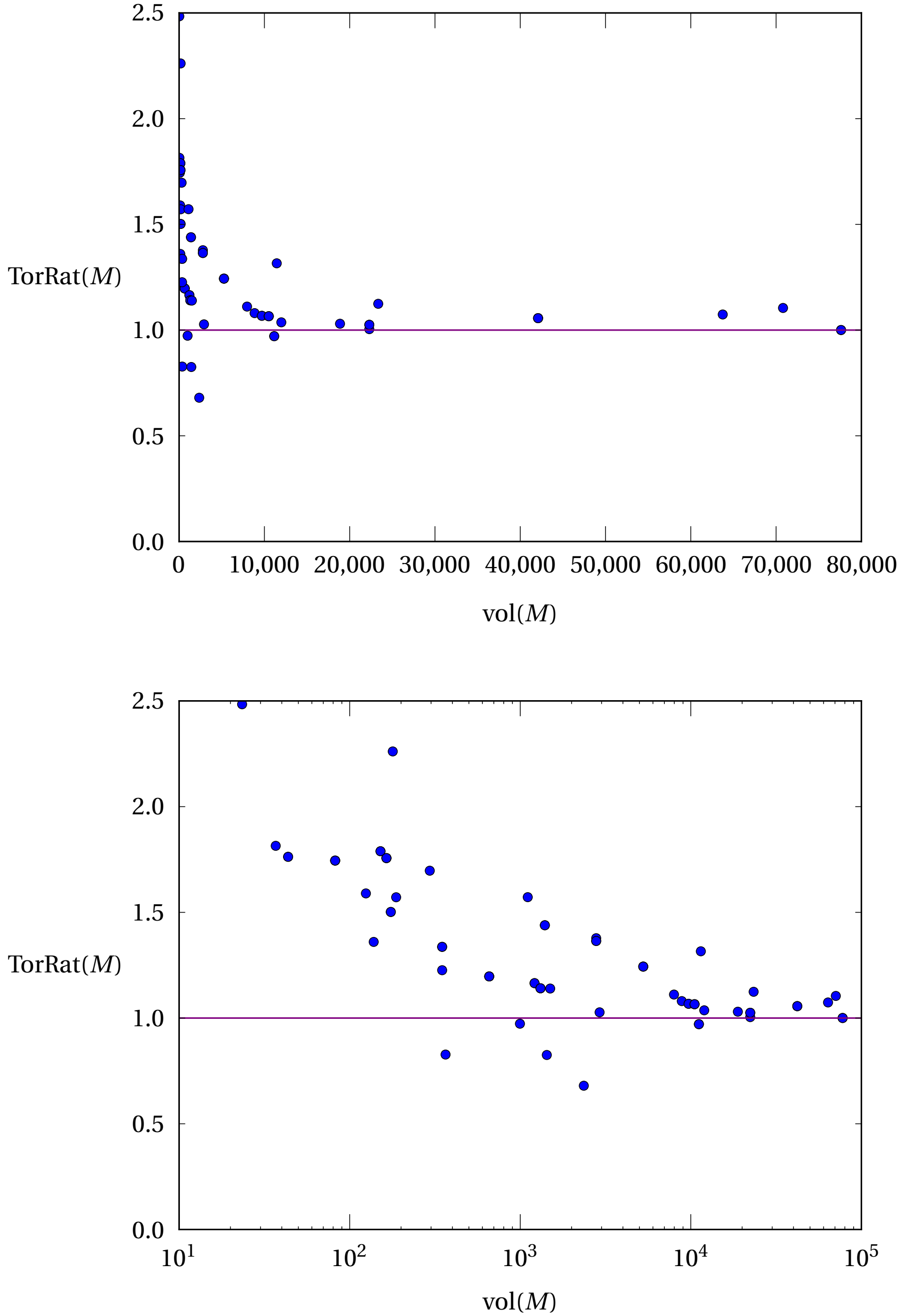


Figure 4.10. Regular congruence covers of level $\mathfrak{p}^n$ where $N(\mathfrak{p}) = 2$. The data is the same in both plots, the only difference being whether the volume axis has a log scale.

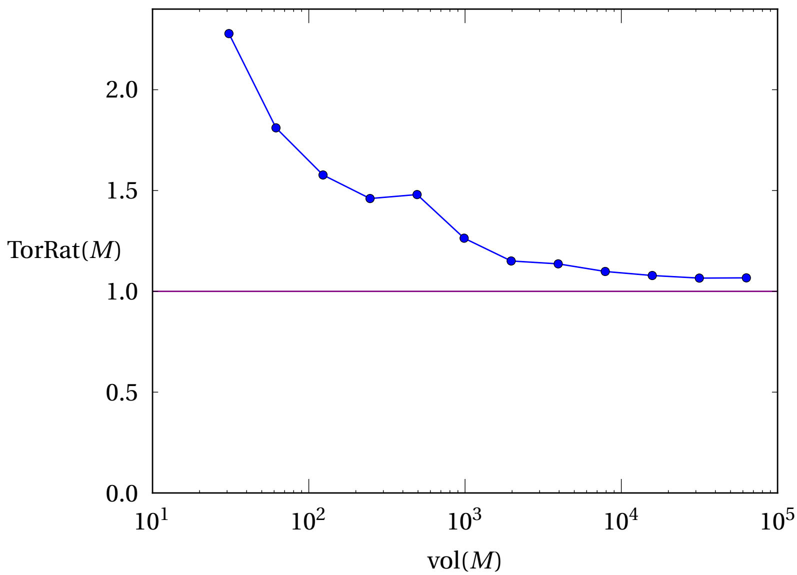


Figure 4.11. The base orbifold M is arithmetic of the following form. The field K has defining polynomial $x^3 + 2x - 1$ and the quaternion algebra D is ramified at the real place of K and the unique prime of norm 4. The orbifold M corresponds to elements of norm one in a maximal order in D. Congruence covers of are type $\Gamma_0(\mathfrak{p}^n)$ where $\mathfrak{p}$ is the prime of norm 2. The values of TorRat in the tail are < 1.07; compare with Figure 4.7.

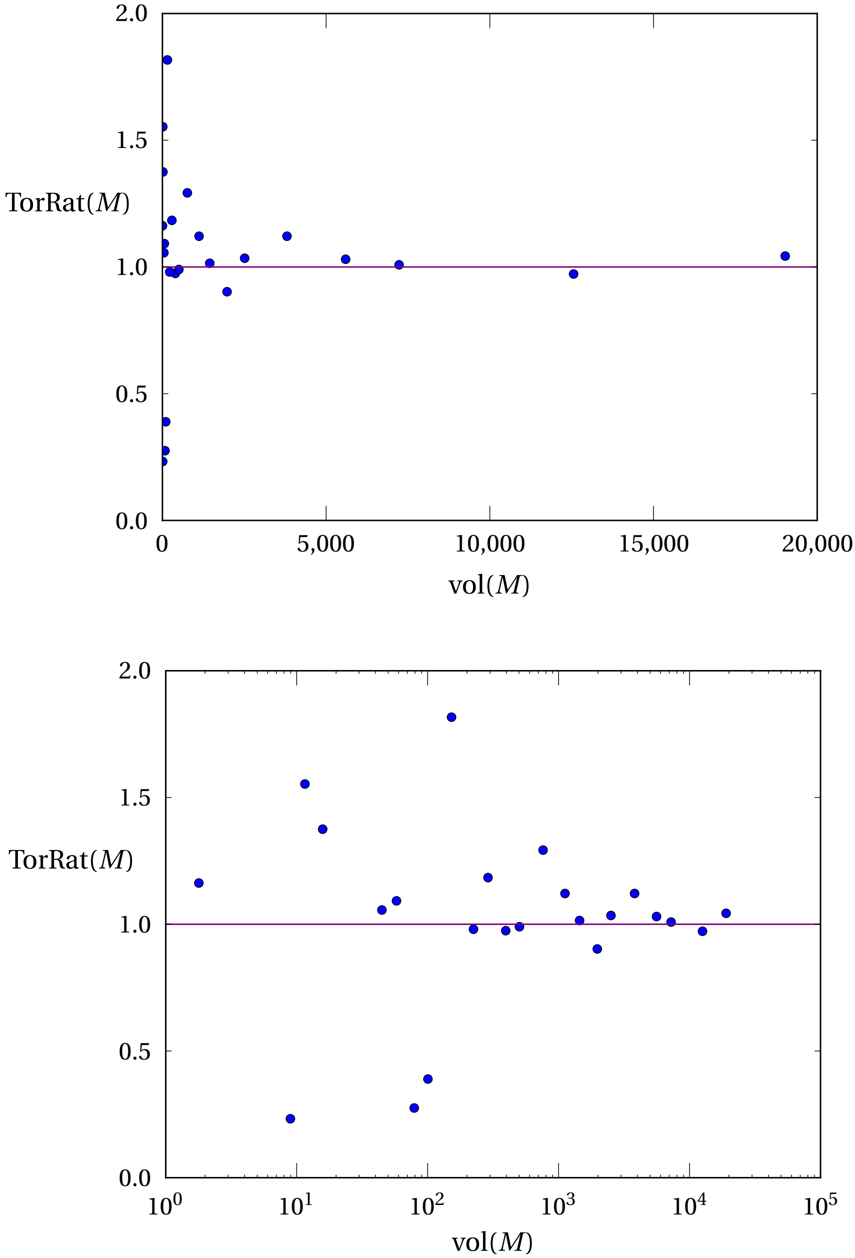
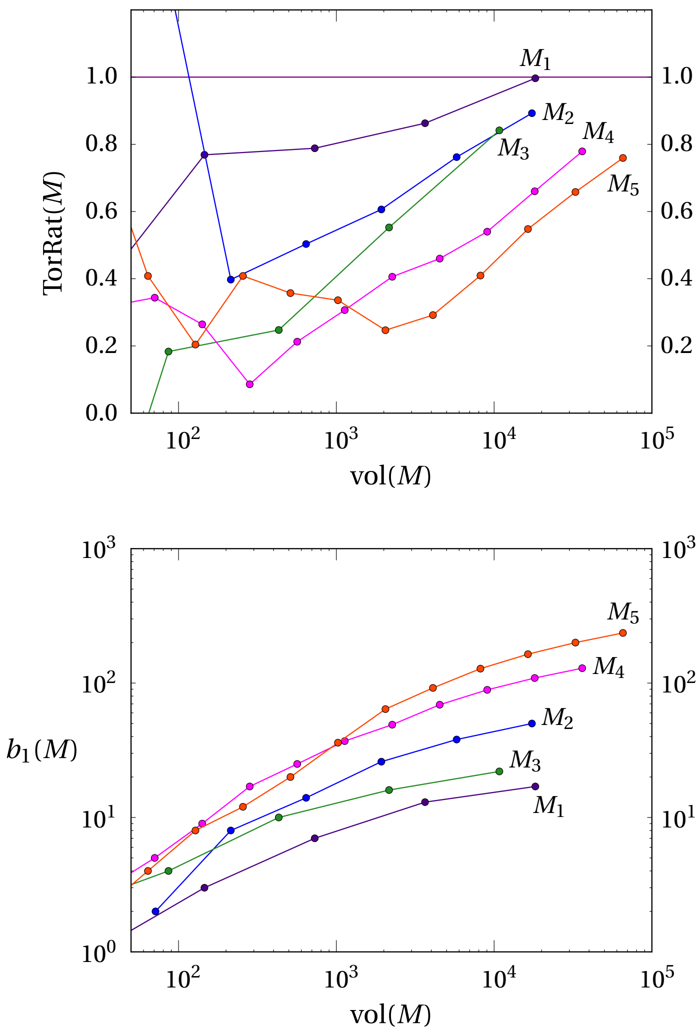


Figure 4.12. Regular congruence covers of level $\mathfrak{p}^n$ where $N(\mathfrak{p}) = 5$. The data is the same in both plots, the only difference being whether the volume axis has a log scale. The base orbifolds come from quaternion algebras over small quartic fields which ramify precisely at the two real places of the base field; all these covers have $b_1 = 0$.



	Defining poly of K	Δ_K	$\mathrm{Ram}_{finite}(D)$	$\mathfrak{p}$	volume
M_1	$x^4 - x^3 - 3x^2 - x + 1$	-1323	$\emptyset$	$\mathfrak{q}_5$	0.9732...
M_2	$x^3 - 2x - 2$	-76	$\{\mathfrak{q}_2\}$	$\mathfrak{q}_3$	0.6617...
M_3	$x^4 - 2x^3 + 3x^2 - 1$	-976	$\emptyset$	$\mathfrak{q}_5$	0.5757...
M_4	$x^3 - x^2 + x - 2$	-83	$\{\mathfrak{q}_5\}$	$\mathfrak{q}_2$	2.9435...
M_5	$x^2 - 7$	-7	$\{\mathfrak{q}_2, \mathfrak{q}_7\}$	$\overline{\mathfrak{q}}_2$	5.3334...

Figure 4.13. Covers of the form $\Gamma_0(\mathfrak{p}^n)$ of the arithmetic orbifolds M_n specified by the data in the table above, specifically the orbifold coming from the elements of norm one in a maximal order of a quaternion algebra D over a field K. Here $\mathfrak{q}_r$ denotes a prime in $\mathcal{O}_K$ of norm r; this prime is unique in every case except the last example, where $\mathfrak{q}_2$ and $\overline{\mathfrak{q}}_2$ denote the two primes in $K = \mathbb{Q}(\sqrt{-7})$ of norm 2.

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